SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 19

MATH 241

14.1.8

Proof. The equation is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$. By separation of variable, you get two ODE's $r^2R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$. By periodicity, $\Theta(\theta + 2\pi) = \Theta(\theta)$. This implies $\lambda = n^2$, and $\Theta = C_1 \cos n\theta + C_2 \sin n\theta, n = 0, 1, 2, \cdots$. The solution to the ODE for R will then be $R = D_1 r^n + D_2 r^{-n}$ for $n > 0$, and $R = D_1 + D_2 \ln r$ for $n = 0$. The solution will have the form

$$u(r,\theta) = (A_0 + B_0 \ln r) + \sum_{n=1}^{\infty}(A_n r^n \cos n\theta + B_n r^{-n} \cos n\theta + C_n r^n \sin n\theta + D_n r^{-n} \sin n\theta)$$

Then let $r = a$ we get $u_0 = A_0 + B_0 \ln a + \sum_{n=1}^{\infty}[(A_n a^n + B_n r^{-n}) \cos n\theta + (C_n b^n + D_n a^{-n}) \sin n\theta]$, so $A_0 + B_0 \ln a = u_0$, $A_n a^n + B_n a^{-n} = 0, C_n a^n + D_n a^{-n} = 0$ for $n > 0$. Let $r = b$ we get $A_0 + B_0 \ln b = u_1$, $A_n a^n + B_n b^{-n} = 0, C_n b^n + D_n b^{-n} = 0$ for $n > 0$.

Combine those equations we get A_n, B_n, C_n, D_n all must vanish for $n > 0$. For $n = 0$, we get $A_0 + B_0 \ln a = u_0, A_0 + B_0 \ln b = u_1$, so $A_0 = \frac{u_0 \ln b - u_1 \ln a}{\ln b - \ln a}$, $B_0 = \frac{u_0 - u_1}{\ln a - \ln b}$. Therefore the solution is

$$u(r,\theta) = \frac{u_0 \ln b - u_1 \ln a}{\ln b - \ln a} + \frac{u_0 - u_1}{\ln a - \ln b}\ln r = \frac{u_0(\ln r - \ln b) - u_1(\ln r - \ln a)}{\ln a - \ln b} = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}$$

□

14.1.11

Proof. The equation is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$. By separation of variable, you get two ODE's $r^2R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$. By the Neumann boundary condition, we have $\Theta'(0) = 0, \Theta'(\frac{\pi}{2}) = 0$. Therefore the eigenvalues are $\lambda = 4n^2, n = 0, 1, 2, \cdots$, and $\Theta = C \cos 2n\theta$. For $n > 0$, $R = C_1 r^n + C_2 r^{-n}$, but it has to be bounded at $r = 0$, thus $C_2 = 0$. For $n = 0$, $R = C_1 + C_2 \ln r$, for the same boundedness reason, $C_2 = 0$. Therefore the solution has the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos 2n\theta$$

Let $r = c$, we get $u(c,\theta) = A_0 + \sum_{n=1}^{\infty} A_n c^n \cos 2n\theta$. Calculate the Fourier cosine series we get $A_0 = \frac{1}{4}, A_n c^n = \frac{\sqrt{2}}{n\pi}$, so $A_n = \frac{\sqrt{2}}{n\pi c^n}$. Therefore the solution is

$$u(r,\theta) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi c^n} r^n \cos 2n\theta$$

□

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Proof. The equation is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$. By separation of variable, you get two ODE's $r^2 R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$. By periodicity, $\Theta(\theta + 2\pi) = \Theta(\theta)$. This implies $\lambda = n^2$, and $\Theta = C_1 \cos n\theta + C_2 \sin n\theta, n = 0, 1, 2, \cdots$. The solution to the ODE for R will then be $R = D_1 r^n + D_2 r^{-n}$ for $n > 0$, and $R = D_1 + D_2 \ln r$ for $n = 0$. For $n > 0$, $R = C_1 r^n + C_2 r^{-n}$, but it has to be bounded at $r = 0$, thus $C_2 = 0$. For $n = 0$, $R = C_1 + C_2 \ln r$, for the same boundedness reason, $C_2 = 0$. Therefore the solution has the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + B_n r^n \sin n\theta$$

Let $r = 1$, we get $f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta$, therefore $A_0 = \frac{1}{2}\frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$, and $A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$, $B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$. Therefore the solution is

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta + \sum_{n=1}^{\infty} \frac{1}{\pi}(\int_0^{2\pi} f(\theta)\cos n\theta d\theta) r^n \cos n\theta + \frac{1}{\pi}(\int_0^{2\pi} f(\theta)\sin n\theta d\theta) r^n \sin n\theta$$

□